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A sketch of brane dynamics in seven and eight dimension using E theory

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Using the general properties that have emerged from E theory we sketch the generic features of the dynamics of branes in seven and eight dimensions. The dynamical equations are a set of duality equations involving the coordinates of the vector representation of E_{11}.

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1. Introduction

The dynamics in E theory follows from the nonlinear realisation of $E_{11} \otimes_s l_1$. The E_{11} part encodes the fields and the vector (l_1) representation encodes the coordinates. The nonlinear realisation is constructed from a group element g which belongs to the group $\mathcal{E}_l$, whose Lie algebra is $E_{11} \otimes_s l_1$, and it is subject to the transformations $g \to g_0 g$ where the rigid transformation $g_0 \in \mathcal{E}_l$ and $g \to gh$ where the local transformation h is in the local subgroup which is specified as part of the definition of the nonlinear realisation. The dynamics is determined by requiring that it is invariant under these two transformations and so different choices of local subalgebra lead to different dynamics. The symmetries in the local subgroup correspond to symmetries that are preserved and so linearly realised, while the ones not in the local subalgebra are those that are spontaneously broken and these are nonlinearly realised.

If we take the fields to depend on the coordinates and the local subalgebra to be the Cartan involution invariant subalgebra of E_{11}, denoted $I_c(E_{11})$ then one derives the low energy effective action for strings and branes as conjectured long ago.[1,2] Indeed, once one restricts to the lowest level fields and coordinates, the dynamics contains the maximal supergravity theories. In particular if one takes the decomposition of E_{11} to $GL(11)$ one finds the equations of motion of 11-dimensional supergravity[3,4] and one will inevitably find the other maximal supergravities if one takes the other decompositions corresponding to the algebra that results from deleting the other nodes in the E_{11} Dynkin diagram (for a review see Ref. 5).

However, if one takes the coordinates to depend on variables that parameterise the branes, the fields to depend on these coordinates and the local subgroup $\mathcal{H}$ to be a subalgebra of $I_c(E_{11})$ then one finds the dynamics of branes. The different choices of local subgroup leads to the different branes. However, each brane carries the full E_{11} symmetry but the symmetries that are spontaneously broken vary from brane to brane.[6,7]

The decomposition of E_{11} to $GL(11)$ results in the theory in 11 dimensions and the generators of E_{11} can be found, for example, in the book,[8] while the generators in the vector representation are given by[2,9,10]

$$\begin{aligned}
&P_{\underline{a}},\ Z^{\underline{a}_1\underline{a}_2},\ Z^{\underline{a}_1\cdots\underline{a}_5},\ Z^{\underline{a}_1\cdots\underline{a}_7,\underline{b}},\ Z^{\underline{a}_1\cdots\underline{a}_8}, Z^{\underline{b}_1\underline{b}_2\underline{b}_3,\underline{a}_1\cdots\underline{a}_8},\\
&\quad Z^{(\underline{c}\,\underline{d}),\underline{a}_1\cdots\underline{a}_9},\ Z^{\underline{c}\,\underline{d},\underline{a}_1\cdots\underline{a}_9},\ Z^{\underline{c},\underline{a}_1\cdots\underline{a}_{10}}\ (2),\ Z^{\underline{a}_1\cdots\underline{a}_{11}},\\
&\quad Z^{\underline{c},\underline{d}_1\cdots\underline{d}_4,\underline{a}_1\cdots\underline{a}_9},\ Z^{\underline{c}_1\cdots\underline{c}_6,\underline{a}_1\cdots\underline{a}_8},\ Z^{\underline{c}_1\cdots\underline{c}_5,\underline{a}_1\cdots\underline{a}_9},\\
&\quad Z^{d_1,\underline{c}_1\underline{c}_2\underline{c}_3,\underline{a}_1\cdots\underline{a}_{10}},\ (2),\ Z^{\underline{c}_1\cdots\underline{c}_4,\underline{a}_1\cdots\underline{a}_{10}},\ (2),\ Z^{(\underline{c}_1\underline{c}_2\underline{c}_3),\underline{a}_1\cdots\underline{a}_{11}},\\
&\quad Z^{\underline{c},\underline{b}_1\underline{b}_2,\underline{a}_1\cdots\underline{a}_{11}},\ (2),\ Z^{\underline{c}_1\cdots\underline{c}_3,\underline{a}_1\cdots\underline{a}_{11}},\ (3),\ldots\,.
\end{aligned} \tag{1.1}$$

Each block of indices contain indices that are totally antisymmetrised except when ( ) is present and this indicates that the indices are symmetrised instead. The elements have multiplicity one except when there is a bracket after the object which contains a number that gives the multiplicity. All the generator belong to irreducible representations of $SL(11)$, for example $Z^{\underline{b}_1\underline{b}_2\underline{b}_3,\underline{a}_1\cdots\underline{a}_8}$ obeys the constraint $Z^{\underline{b}_1\underline{b}_2[\underline{b}_3,\underline{a}_1\cdots\underline{a}_8]} = 0$.

The theory in D dimensions results from deleting the node D in the E_{11} Dynkin diagram and the generators in the vector representation which have totally antisymmetrised indices are given in Table 1.[6,10,11]

We note the presence of the generators at level one that are Lorentz scalars but belong to representations of E_{11-D}. Given the one-to-one relation between generators in the vector representation and coordinates in the nonlinear realisation, it follows that for each element in the table we have coordinates in the space–time through which the brane moves. As such the coordinates in the above table are the main characters in this paper. In fact, there are an infinite number of coordinates in the vector representation most of which have indices that cannot be written as a single antisymmetrised block. These later coordinates will not play a significant role in this paper as we will be concerned with low level branes. The coordinates in the vector representation at higher levels than the usual coordinates x^a of our familiar space–time play an essential role in the construction of the low energy effective action of strings and branes even though they must be truncated out to gain the supergravity results we are familiar with. The Lorentz scalar coordinates, in the first column first proposed in the E_{11} papers referenced above, are the starting point for papers on the so-called exceptional field theory, see for example Ref. 12 for an

Table 1. The form generators in the l_1 representation in D dimensions.

D	G	Z	$Z^{\underline{a}}$	$Z^{\underline{a}_1\underline{a}_2}$	$Z^{\underline{a}_1\cdots\underline{a}_3}$	$Z^{\underline{a}_1\cdots\underline{a}_4}$	$Z^{\underline{a}_1\cdots\underline{a}_5}$	$Z^{\underline{a}_1\cdots\underline{a}_6}$	$Z^{\underline{a}_1\cdots\underline{a}_7}$
8	$SL(3)\otimes SL(2)$	$(\mathbf{3},\mathbf{2})$	$(\bar{\mathbf{3}},\mathbf{1})$	$(\mathbf{1},\mathbf{2})$	$(\mathbf{3},\mathbf{1})$	$(\bar{\mathbf{3}},\mathbf{2})$	$(\mathbf{1},\mathbf{3})$	$(\mathbf{3},\mathbf{2})$	$(\mathbf{6},\mathbf{1})$
							$(\mathbf{8},\mathbf{1})$	$(\bar{\mathbf{6}},\mathbf{2})$	$(\mathbf{18},\mathbf{1})$
							$(\mathbf{1},\mathbf{1})$		$(\mathbf{3},\mathbf{1})$
									$(\mathbf{6},\mathbf{1})$
									$(\mathbf{3},\mathbf{3})$
7	$SL(5)$	$\mathbf{10}$	$\bar{\mathbf{5}}$	$\mathbf{5}$	$\overline{\mathbf{10}}$	$\mathbf{24}$	$\mathbf{40}$	$\mathbf{70}$	—
						$\mathbf{1}$	$\mathbf{15}$	$\mathbf{50}$	—
							$\mathbf{10}$	$\mathbf{45}$	—
								$\mathbf{5}$	—
6	$SO(5,5)$	$\overline{\mathbf{16}}$	$\mathbf{10}$	$\mathbf{16}$	$\mathbf{45}$	$\overline{\mathbf{144}}$	$\mathbf{320}$	—	—
					$\mathbf{1}$	$\mathbf{16}$	$\mathbf{126}$	—	—
							$\mathbf{120}$	—	—
5	E_6	$\overline{\mathbf{27}}$	$\mathbf{27}$	$\mathbf{78}$	$\overline{\mathbf{351}}$	$\mathbf{1728}$	—	—	—
				$\mathbf{1}$	$\overline{\mathbf{27}}$	$\mathbf{351}$	—	—	—
						$\mathbf{27}$	—	—	—
4	E_7	$\mathbf{56}$	$\mathbf{133}$	$\mathbf{912}$	$\mathbf{8645}$	—	—	—	—
			$\mathbf{1}$	$\mathbf{56}$	$\mathbf{1539}$	—	—	—	—
					$\mathbf{133}$	—	—	—	—
					$\mathbf{1}$	—	—	—	—
3	E_8	$\mathbf{248}$	$\mathbf{3875}$	$\mathbf{147250}$	—	—	—	—	—
		$\mathbf{1}$	$\mathbf{248}$	$\mathbf{30380}$	—	—	—	—	—
			$\mathbf{1}$	$\mathbf{3875}$	—	—	—	—	—
				$\mathbf{248}$	—	—	—	—	—
				$\mathbf{1}$	—	—	—	—	—

account of this. As we will discover, while some coordinates given the embedding of the brane in our usual space–time, some of the coordinates are the worldvolume fields of the branes.[6,7]

We now briefly review some of the main features of how one computes the brane dynamics from the Cartan forms.[6,7] While this paper does not contain any detailed calculations the construction below is required to justify the discussions in this paper. We can write the group element g of the nonlinear realisation in the form $g = g_l g_h g_E$ where g_E is in the Borel subgroup of E_{11}, g_l is formed from the generators of the l_1 representation and g_h belongs to $I_c(E_{11})$. These group elements can be written in the form

$$g_l = e^{z^A l_A}\,, \quad g_E = e^{A_{\bar{\alpha}} R^{\bar{\alpha}}}\,, \quad g_h = e^{\varphi\cdot S}\,, \tag{1.2}$$

where $R^{\bar{\alpha}}$ and $S^{\bar{\alpha}}$are the generators of the Borel subalgebra of E_{11} and $I_c(E_{11})$ respectively. In Eq. (1.2), z^A are the coordinates of the background space–time and they depend on the parameters $\xi^{\underline{\alpha}}$ of the brane worldvolume. The fields $A_{\underline{\alpha}}$ are the E_{11} background fields, which include those of the maximal supergravity theories, and they depend on the coordinates of the background space–time z^A. The fields φ also depend on $\xi^{\underline{\alpha}}$. The transformation h of the local subgroup $\mathcal{H}$ depends on the parameters $\xi^{\underline{\alpha}}$ in an arbitrary way and we can use this to set some of the fields φ to zero. We note that the nonlinear realisation used for branes involves the additional fields φ which are not present for the nonlinear realisation used to derive the low energy effective action of strings and branes.

The Cartan forms are given by

$$\mathcal{V} = g^{-1}\, dg = \mathcal{V}_E + \mathcal{V}_l + \mathcal{V}_h^B \,, \tag{1.3}$$

where

$$\begin{gathered} \mathcal{V}_E = g_E^{-1}\, dg_E \quad \text{and} \quad \mathcal{V}_l = g_E^{-1} g_h^{-1}\big(g_l^{-1}\, dg_l\big) g_h g_E \,, \\ \mathcal{V}_h^B = g_E^{-1}\big(g_h^{-1}\, dg_h\big) g_E = g_E^{-1}\mathcal{V}_h g_E = g_E^{-1} g_h^{-1}\, dg_h g_E \,. \end{gathered} \tag{1.4}$$

The Cartan forms $\mathcal{V}_E$ are just the Cartan forms of E_{11} and they only depend on the background fields $A_{\underline{\alpha}}$. The Cartan forms associated with the vector representation are given by

$$\begin{aligned} \mathcal{V}_l \equiv \nabla^B z^A l_A &= g_E^{-1} g_h^{-1}\big(dz^A l_A\big) g_h g_E \\ &= g_E^{-1}\big(\nabla z^A l_A\big) g_E \equiv \nabla z^\Pi E_\Pi{}^A l_A \,, \end{aligned} \tag{1.5}$$

where $\nabla \equiv d\xi^{\underline{\alpha}}\nabla_{\underline{\alpha}}$ and $E_\Pi{}^A$ is defined by $g_E^{-1}\, dz \cdot l g_E \equiv dz^\Pi\, E_\Pi{}^A l_A$ which is the vielbein in background space–time and also only depends on the background fields $A_{\underline{\alpha}}$. The fields φ only occur in the Cartan forms $\nabla_{\underline{\alpha}} z^A$ and $g_h^{-1} dg_h$ which are independent of the background fields $A_{\underline{\alpha}}$.

The Cartan forms are inert under the rigid g_0 transformations, but under the local $h \in \mathcal{H}$ transformations they transform as

$$\mathcal{V} \to h^{-1}\mathcal{V}h + h^{-1}\, dh \tag{1.6}$$

and in particular that

$$\nabla^B z^A l_A \to h^{-1}\big(\nabla^B z^A l_A\big)h \,, \quad \mathcal{V}_h^B \to h^{-1}\mathcal{V}_h^B h + h^{-1}\, dh \,. \tag{1.7}$$

Using this equation it is straightforward to explicitly compute the local transformations of the individual Cartan forms using the $E_{11} \otimes_s l_1$ algebra. As the dynamics consists of a set of equations that are invariant under these transformations and that the $\nabla_{\underline{\alpha}}^B z^A$ transform covariantly, we are looking for equations which relate these Cartan forms to each other. We will also demand that the equations are invariant under arbitrary reparameterisations of the brane worldvolume.

For simplicity we will consider the brane dynamics in the absence of background fields. In this case we will take g_E to be the identity element and instead of taking the algebra $E_{11} \otimes_s l_1$ we take the nonlinear realisation of $I_c(E_{11}) \otimes_s l_1$ with local subalgebra $\mathcal{H}$. Then we only have the Cartan forms $\nabla_{\underline{\alpha}} z^A$ and $\mathcal{V}_h = g_h^{-1} dg_h$. In fact, the equations of motion are constructed from only the Cartan forms $\nabla_{\underline{\alpha}} z^A$.

The brane dynamics in the presence of the background fields can be readily found from the resulting equations by using Eq. (1.5) to simply reinstate their presence by introducing the vielbein in the way that this equation dictates, that is, make the replacement $\nabla_{\underline{\alpha}} z^A \to \nabla_{\underline{\alpha}}^B z^A$. We will not in this paper consider the construction of the Wess–Zumino term in the brane dynamics.

Although, there are still a number of features of the above construction which are yet to be fully understood, the general features are apparent.[6,7]

- The equations of motion are constructed from objects that are first order in derivatives, that is, they involve the Cartan forms $\nabla_{\underline{\alpha}} z^A$ and not derivatives acting on these forms. As a result they are equations which equate the different Cartan forms associated with the vector representation to each other in such a way as to preserve the local subalgebra $\mathcal{H}$.
- Some of the equations which follow from the nonlinear realisation are ones which can be used to solve analytically for all of the fields φ in terms of the coordinates, while others are dynamical equations for the coordinates and these are duality conditions.

In this paper, we will use the above guidelines to sketch the dynamics of the low level branes in seven and eight dimensions. The advantage of this approach is that the reader can see what are the general features of branes in E theory without being distracted by the E_{11} formalism and a morass of equations. We will, in particular, focus on finding the generic form of the dynamical equations rather than the equations that are used to solve algebraically for the fields φ.

As just discussed we are searching for duality relations between the Cartan forms which are invariant under the transformations of the linear local subalgebra $\mathcal{H}$. We need not consider the rigid transformations as the Cartan forms are invariant under them. At the linearised level in the fields the equations of motion must be linear in the Cartan forms and as such we begin by searching for equations which relate one Cartan form to another. Indeed we must find relations that pair up the Cartan forms and relate them using the epsilon symbol on the worldvolume of the brane. For a p-brane we will divide the indices $\underline{a}$, $\underline{b}, \ldots = 0, 1, \ldots, D-1$, where D is the dimension of the space–time in which the brane moves, into those in the brane directions a, $b, \ldots = 0, 1, \ldots, p$ and those transverse to the brane a', $b', \ldots = p+1, \ldots, D-1$. As such the epsilon symbol in the brane worldvolume is denoted by $\epsilon^{a_1 \cdots a_{p+1}}$. We will adopt this convention throughout this paper.

We recall that the theory in D dimensions arises when we delete the node labelled D in the E_{11} Dynkin diagram which leaves the algebra $GL(D) \otimes E_{11-D}$. We then decompose E_{11} into representations of this later algebra. These representations are labelled by a level which depends on the node being deleted; the level zero representation is just the algebra that remains when we delete the node labelled D in the E_{11} Dynkin, namely $GL(D) \otimes E_{11-D}$. The Cartan involution invariant subalgebra of E_{11}, denoted $I_c(E_{11})$, is of the form $R^\alpha - R^{-\alpha}$ where α is a positive root which has a positive level. Thus the generators in $I_c(E_{11})$ are made up of two generators of opposite level. The exception is when these generators have level zero. We will refer to this as the level zero part of $I_c(E_{11})$ and it is the Cartan involution invariant subalgebra of $GL(D) \otimes E_{11-D}$ which contains the Lorentz algebra $SO(1, D-1)$ and $I_c(E_{11-D})$ and is just $SO(1, D-1) \otimes I_c(E_{11-D})$.

The nonlinear realisation used to construct the low energy effective action for string and branes requires a local subalgebra $\mathcal{H}$ which is a subalgebra of $I_c(E_{11})$. Indeed, a p-brane breaks the $SO(1, D-1)$ Lorentz symmetry of the background space–time to $SO(1,p) \otimes SO(D-p-1)$ which is part of $\mathcal{H}$. A brane charge also generically belongs to a representation of the U duality algebra, E_{11-D}, and so also to $I_c(E_{11-D})$. As such the particular brane charge which is active will also break $I_c(E_{11-D})$ into an algebra that is part of $\mathcal{H}$ and which we denote by $\mathcal{H}_0^U$. If we denote the level zero part of $\mathcal{H}$, in the above sense, by $\mathcal{H}_0$, then $\mathcal{H}_0$ will contain $SO(1,p) \otimes SO(D-p-1)$ and $\mathcal{H}_0^U$; indeed $\mathcal{H}_0 = SO(1,p) \otimes SO(D-p-1) \otimes \mathcal{H}_0^U$.

A brane moves through a space–time with the coordinates in the vector representation. In any dimensions the level zero coordinate in the vector representation is $x^{\underline{a}}$ which is the usual coordinate of our familiar space–time. As we have just explained this must satisfy a duality relation with one of the other coordinates. If we consider a simple p-brane, that is, a brane whose charge in totally antisymmetric in its indices, then its charge will be of the form $Z^{a_1 \cdots a_{p+1} \bullet}$ where $\bullet$ denotes the indices of the representation of the U duality group E_{11-D} to which it belongs. This representation breaks into representations of $I_c(E_{11-D})$. The particular brane charge that is active will break this latter group to $\mathcal{H}_0^U$ under which the active charge is a singlet. Corresponding to this singlet charge the nonlinear realisation possess a coordinate which we denote by $y_{\underline{a}_1 \cdots \underline{a}_{p+1}}$. We can write down a duality relation between this coordinate and the coordinate[7] $x^{\underline{a}}$

$$\nabla_\alpha x^a = -e_1 \epsilon^{a b_1 \cdots b_p} \nabla_\alpha y_{b_1 \cdots b_p} \,, \tag{1.8}$$

where e_1 is a constant. In all the cases studied so far one also finds that the nonlinear realisation implies the condition $\nabla_\alpha x^{a'} = 0$. Part of this equation can be solved for a certain field φ and the other part is a dynamical equation for the transverse coordinates $x^{a'}$.[7] As a result one can derive the equation

$$\sqrt{-\gamma}\gamma^{\alpha\beta}\nabla_\beta x^{\underline{a}} = e_1 \epsilon^{\alpha\beta\gamma_1\cdots\gamma_{p-1}} \nabla_\beta x^{\underline{a}\underline{b}_1\cdots\underline{b}_{p-1}} \nabla_{\gamma_1} x_{\underline{b}_1} \cdots \nabla_{\gamma_{p-1}} x_{\underline{b}_{p-1}} \,, \tag{1.9}$$

where $\gamma_{\alpha\beta} = \nabla_\alpha x^a \nabla_\beta x^b \eta_{ab}$. This last equation can be shown to be the familiar equations for brane dynamics and we refer the reader to Ref. 7 for an account. These equations hold for all the branes considered in this paper and so we will in what follows concentrate on the other dynamical equations.

The higher level symmetries of the nonlinear realisation will transform Eq. (1.9) into equations that are duality relations between the Cartan forms of some the other coordinates in the vector representation. However, in this paper, rather than carry out such transformations we will simply search for generic duality relations that are invariant under the local transformations of level zero, $\mathcal{H}^0$, which consists of the groups $SO(1,p) \otimes SO(D-p-1)$ and $\mathcal{H}_0^U$. As it is a duality relation, it will contain the epsilon symbol $\epsilon^{a_1\cdots a_{p+1}}$. Let us consider the coordinates $x_{a_1\cdots a_n}$ which will have the Cartan form $\nabla_\alpha x_{a_2\cdots a_{n+1}}$. It will prove advantageous to rewrite this Cartan as $\nabla_{a_1} x_{a_2\cdots a_{n+1}}$ where $\nabla_a = (s^{-1})_a{}^\alpha \nabla_\alpha$ and $s_\alpha{}^a = \nabla_\alpha x^a$. We will also use this definition throughout this paper. We expect that it will be related to a coordinate with m indices, that is $x_{a_1\cdots a_m}$ where $m = p-n-1$ through an equation whose generic form is given by

$$\nabla_{a_1} x_{a_2\cdots a_{n+1}} = e_2 \epsilon_{a_1 a_2 \cdots a_{n+1}}{}^{b_1 b_2 \cdots b_{m+1}} \nabla_{b_1} x_{b_2\cdots b_{m+1}}\,, \tag{1.10}$$

where e_2 is a constant.

The above discussion has neglected the fact that the coordinates $x_{a_1\cdots a_n}$ and $x_{a_1\cdots a_m}$ belong to the representations, denoted by R_n^0 and R_m^0 respectively of the U duality group. However, these representations decompose into a sum of representations of the algebra $\mathcal{H}_0^U$ preserved by the brane. To find a consistent equation one must select representations in the two decompositions that can be related by an $\mathcal{H}_0^U$ invariant tensor and use this tensor in the above duality relation. If $m = n$, that is when $2n = p-1$ the duality relation can become a self-duality relation.

The dynamics of the two and five brane in 11 dimensions, from the viewpoint of E theory, was given in Refs. 6 and 7. It will be instructive to illustrate how the above procedures apply to these branes. The brane charges in eleven dimensions are given in Eq. (1.2) and as a result the nonlinear realisation has the following coordinates:

$$\begin{aligned}
&x^{\underline{a}},\ x_{\underline{a}_1\underline{a}_2},\ x_{\underline{a}_1\cdots\underline{a}_5},\ x_{\underline{a}_1\cdots\underline{a}_7,\underline{b}},\ x_{\underline{a}_1\cdots\underline{a}_8},\ x_{\underline{b}_1\underline{b}_2\underline{b}_3,\underline{a}_1\cdots\underline{a}_8},\\
&\quad x_{(\underline{c}\underline{d}),\underline{a}_1\cdots\underline{a}_9},\ x_{\underline{c}\underline{d},\underline{a}_1\cdots\underline{a}_9},\ x_{\underline{c},\underline{a}_1\cdots\underline{a}_{10}}\ (2),\ x_{\underline{a}_1\cdots\underline{a}_{11}},\\
&\quad x_{\underline{c},\underline{d}_1\cdots\underline{d}_4,\underline{a}_1\cdots\underline{a}_9},\ x_{\underline{c}_1\cdots\underline{c}_6,\underline{a}_1\cdots\underline{a}_8},\ x_{\underline{c}_1\cdots\underline{c}_5,\underline{a}_1\cdots\underline{a}_9},\\
&\quad x_{\underline{d}_1,\underline{c}_1\underline{c}_2\underline{c}_3,\underline{a}_1\cdots\underline{a}_{10}},\ (2), x_{\underline{c}_1\cdots\underline{c}_4,\underline{a}_1\cdots\underline{a}_{10}},\ (2),\ x_{(\underline{c}_1\underline{c}_2\underline{c}_3),\underline{a}_1\cdots\underline{a}_{11}},\\
&\quad x_{\underline{c},\underline{b}_1\underline{b}_2,\underline{a}_1\cdots\underline{a}_{11}},\ (2),\ x_{\underline{c}_1\cdots\underline{c}_3,\underline{a}_1\cdots\underline{a}_{11}},\ (3),\ldots\,.
\end{aligned} \tag{1.11}$$

The Cartan involution invariant subalgebra of E_{11}, denoted $I_c(E_{11})$, has the Lorentz algebra $SO(1,10)$ at level zero.

The charge of the M2 brane is $Z^{a_1a_2}$ and selecting a particular charge breaks the $SO(1,10)$ Lorentz algebra down to $SO(1,2)\otimes SO(8)$ which is the local subalgebra $\mathcal{H}$ at level zero, that is $\mathcal{H}_0 = SO(1,2)\otimes SO(8)$. Corresponding to the M2 brane charge we have the coordinate $x_{a_1a_2}$. This coordinate, together with the coordinate x^a will obey the duality relation of Eq. (1.9). This does indeed correctly describe the motion of the M2 brane.

The five brane has the brane charge $Z^{a_1\cdots a_5}$ and this brane breaks the Lorentz symmetry down to $SO(1,5)\otimes SO(5)$. The corresponding coordinate $x_{a_1\cdots a_5}$, together with the coordinate x^a obeys Eq. (1.9). At a lower level we have the two form coordinate $x_{a_1a_2}$ which, following the above discussion, should satisfy a self-duality relation which is invariant under the level zero symmetries, that is,

$$\nabla_{[a_1}x_{a_2a_3]} = \frac{1}{3!}\epsilon_{a_1a_2a_3}{}^{b_1b_2b_3}\nabla_{b_1}x_{b_2a_3}\,, \tag{1.12}$$

where ∇_a was defined by Eq. (1.10). These equations we derived in Ref. 7 using the higher level symmetries in addition to those at level zero. They reproduce the correct dynamics to all orders in the usual embedding coordinate x^a and up to the linear level in the worldvolume field $x_{a_1a_2}$; the situation for nonlinear terms in the later field is discussed in Ref. 7. These branes illustrate the general pattern, the coordinates in the vector representation contain the usual embedding coordinate as well as the worldvolume fields and they satisfy duality relations.

2. Branes in Seven Dimensions

The seven-dimensional theory emerges when we decompose E_{11} into $GL(7)\otimes SL(5)$ which is the algebra that emerges when we delete node seven in the E_{11} Dynkin diagram. The generators in the vector representation are given by[7]

$$\begin{aligned} &P_{\underline{a}};\ Z^{MN};\ Z^{\underline{a}}{}_M;\ Z^{\underline{a}_1\underline{a}_2M};\ Z^{\underline{a}_1\underline{a}_2\underline{a}_3}{}_{MN};\\ &\qquad Z^{\underline{a}_1\underline{a}_2\underline{a}_3,\underline{b}},\ Z^{\underline{a}_1\cdots\underline{a}_4},\ Z^{\underline{a}_1\cdots\underline{a}_4M}{}_N,\ Z^{\underline{a}_1\cdots\underline{a}_5MN},\\ &\qquad Z^{\underline{a}_1\cdots\underline{a}_5(MN)},\ Z^{\underline{a}_1\cdots\underline{a}_5}{}_{MN,P},\ Z^{\underline{a}_1\cdots\underline{a}_4,\underline{b}MN},\ldots\,, \end{aligned} \tag{2.1}$$

where $\underline{a}, \underline{b},\ldots = 0,1,\ldots,6$ and the indices $M, N,\ldots = 1,\ldots,5$ are those of $SL(5)$.

As a result the brane moves through a space–time with the coordinates

$$x^{\underline{a}};\ x_{MN};\ x_{\underline{a}}{}^M;\ x_{\underline{a}_1\underline{a}_2M};\ x_{\underline{a}_1\underline{a}_2\underline{a}_3}{}^{MN};\ x_{\underline{a}_1\underline{a}_2\underline{a}_3,\underline{b}};\ x_{\underline{a}_1\cdots\underline{a}_4};\ x_{\underline{a}_1\cdots\underline{a}_4M}{}^N;\ldots \tag{2.2}$$

and the Cartan forms which belong to the vector representation of the E_{11} algebra can be written in the form

$$\begin{aligned} \mathcal{V}_l &= \nabla x^{\underline{a}}P_{\underline{a}} + \nabla x_{PQ}Z^{PQ} + \nabla x_{\underline{a}}{}^M Z^{\underline{a}}{}_M + \nabla x_{\underline{a}_1\underline{a}_2M}Z^{\underline{a}_1\underline{a}_2M}\\ &\quad + \nabla x_{\underline{a}_1\underline{a}_2\underline{a}_3}{}^{MN}Z^{\underline{a}_1\underline{a}_2\underline{a}_3}{}_{MN} + \nabla x_{\underline{a}_1\underline{a}_2\underline{a}_3,b}Z^{\underline{a}_1\underline{a}_2\underline{a}_3,\underline{b}}\\ &\quad + \nabla x_{\underline{a}_1\cdots\underline{a}_4}Z^{\underline{a}_1\cdots\underline{a}_4} + \nabla x_{\underline{a}_1\cdots\underline{a}_4M}{}^N Z^{\underline{a}_1\cdots\underline{a}_4M}{}_N + \cdots\,. \end{aligned} \tag{2.3}$$

The Cartan forms transform under the local subalgebra $\mathcal{H}$ which is a subalgebra of $I_c(E_{11})$. At level zero the latter contains the Lorentz algebra $SO(1,6)$ and the U duality algebra $SO(5)$ and as such at level zero $\mathcal{H}$ is a subalgebra of this. We have already constructed the dynamics of the one and two branes in seven dimensions but it will be useful to illustrate the methods of this paper to find their generic form.

2.1. *The one brane*

The one brane has a two-dimensional worldvolume and, as explained above, we take the indices in the directions of the worldvolume of the string to take the values a,

$b, \ldots = 0, 1$ and the reminder to be given by a', $b', \ldots = 2, \ldots, 6$. The brane will preserve $SO(1,1) \otimes SO(5)$ of the $SO(1,6)$ Lorentz symmetry and so this symmetry belongs to the local subaglebra $\mathcal{H}$. The one brane charge is given by $Z_M^{\underline{a}}$. A given string has a given charge and making this selection we break the internal $SO(5)$ symmetry down to $SO(4)$ which will also belong to $\mathcal{H}$. Indeed the level zero part is given by $\mathcal{H}_0 = SO(1,1) \otimes SO(5) \otimes SO(4)$. As explained at the end of Sec. 1, we adopt the duality relation (1.9) between the corresponding coordinate $y_{\underline{a}}$ and the usual space–time coordinate $x^{\underline{a}}$.

Examining the coordinates of Eq. (2.1) we find the Lorentz scalar coordinates x_{PQ} which under the decomposition to $SO(4)$ leads to a coordinates in the $10 = 6 \oplus 4$ representations of $SO(4)$. We can denote these by x_{ij} and x_i, where $i, j = 1, \ldots, 4$, respectively. Taking the former coordinate, we can write down equations that are first order in the Cartan forms and are invariant under the local subalgebra $\mathcal{H}$ and in particular its level zero part, given above. Such equations are of the generic form

$$\nabla_a x_{ij} = -\frac{1}{2}\epsilon_a{}^b \epsilon_{ijkl} \nabla_b x^{kl}$$

or equivalently

$$\sqrt{-\gamma}\gamma^{\alpha\beta}\nabla_\beta x_{ij} = -\frac{1}{2}\epsilon^{\alpha\beta}\epsilon_{ijkl}\nabla_\beta x^{kl}\,, \tag{2.4}$$

where $\nabla_a = (s^{-1})_a{}^\alpha \nabla_\alpha$ and $s_\alpha{}^a = \nabla_\alpha x^a$. Of course to find the full equations of motion one has to find a set of equations that are invariant under the full symmetries of the nonlinear realisation and not just the level zero symmetries. This has been done in Ref. 7, where the choice of local subalgebra $\mathcal{H}$ and the corresponding transformations of the Cartan forms can be found.

Assuming that the equations that follow from the nonlinear realisation imply that these are the only dynamical field we can count the number of bosonic degrees of freedom. We have $7 - 2 = 5$ degrees of freedom in $x^{a'}$, taking into account the worldvolume reparameterisation symmetry, and $\frac{4.3}{2.2} = 3$ from x_{ij} which gives us 8 bosonic degrees of freedom. This is the number required for a half BPS brane that is maximally supersymmetric. A brane with these degrees of freedom would arise from the dimensional reduction of the IIA string.

2.2. *The two brane*

The two brane has a three-dimensional worldvolume and it breaks the Lorentz symmetry $SO(1,6)$ into $SO(1,2) \otimes SO(4)$ which is in the local subalgebra. The charge for the two brane is $Z^{a_1 a_2 M}$ which transforms in the 5 of the internal $SO(5)$ symmetry. Selecting a particular charge breaks $SO(5)$ to $SO(4)$. As a result the local subalgebra at level zero is $\mathcal{H}_0 = SO(1,2) \otimes SO(4) \otimes SO(4)$. The active two brane charge corresponds to a coordinate which we denote by $y^{\underline{a}_1 \underline{a}_2}$ and this, together with the coordinate $x^{\underline{a}}$, will satisfy the duality relation of Eq. (1.9).

Examining the other coordinates in Eq. (2.2) we find the coordinates x_{MN} and $x_{\underline{a}}{}^M$ which, under the decomposition to $SO(4)$, decompose into a $10 = 6 \oplus 4$ and a $5 = 4 \oplus 1$ respectively. To find a duality relation between the corresponding Cartan forms we must choose the 4 from each coordinate and then we can write down the relation

$$\nabla_a x_i + \frac{1}{4}\epsilon_a{}^{c_1 c_2}\nabla_{c_1} x_{c_2 i} = 0$$

or equivalently

$$\sqrt{-\gamma}\,\gamma^{\alpha\beta}\nabla_\beta x_i + \frac{1}{4}\epsilon^{\alpha\beta_1\beta_2}\nabla_{\beta_1} x_{di}\nabla_{\beta_2} x^d = 0\,. \tag{2.5}$$

Assuming that these are the only active fields the number of bosonic degrees of freedom are 4 for $x^{a'}$ and 4 for x_i giving a total of 8 bosonic degrees of freedom. This is the correct number for a maximally supersymmetric brane in a type II theory. This is the same content as the dimensional reduced M2 brane of eleven dimensions. The fully nonlinear equations of motion, the choice of local subalgebra $\mathcal{H}$ and the corresponding transformations of the Cartan forms were given in Ref. 7.

2.3. *The three brane*

The three brane has a four-dimensional worldvolume and the part of the Lorentz symmetry $SO(1,6)$ which is in the local subalgebra $\mathcal{H}$ is $SO(1,3) \otimes SO(3)$. The charge for the three brane is $Z^{\underline{a}_1\underline{a}_2\underline{a}_3 MN}$ which belongs to the ten-dimensional representation of the internal symmetry $SO(5)$. Choosing a particular charge for the three brane preserves only $SO(3)\otimes SO(2)$ which belongs to the local subalgebra $\mathcal{H}$ at level zero. Decomposing the 10 into this group we find that it consists of $10 = (3,1) \oplus (3,2) \oplus (1,1)$, the last component being the active three brane charge. Let us denote the coordinate corresponding to this charge by $y_{\underline{a}_1\underline{a}_2\underline{a}_3}$ and this, together with the coordinate x^a, can be taken to obey the duality equation of Eq. (1.9).

Examining Eq. (2.2) we find the coordinate $x_a{}^M$ which belongs to the 5 of $SO(5)$. which can only be dual to itself given that the worldvolume epsilon symbol has four space–time indices. The 5 decomposes into the $5 = (3,1) \oplus (1,2)$ representations of $SO(3) \otimes SO(2)$. If we consider the latter we can write down the duality equation

Please check if this is relation?

$$\nabla_{[a_1} x_{a_2]}{}^{i'} = \pm\frac{1}{2}\epsilon_{a_1 a_2}{}^{b_1 b_2}\epsilon^{i'j'}\nabla_{b_1} x_{b_2 j'}\,, \tag{2.6}$$

where i', j', $\ldots = 1, 2$ are the $SO(2)$ indices. There are no consistent duality relations one can write down for the $(3,1)$.

The coordinate x_{MN} of Eq. (2.1) could be dual to the coordinate $x_{a_1 a_2 M}$ which belong to the 10 and 5 of $SO(5)$, respectively. Using the decompositions of these representations given above we find that they have $(3,1)$ in common which we denote by x_{ij} and $x_{a_1 a_2 i}$ where $i, j, \ldots = 1, 2, 3$, respectively. Using these coordinates we can write down the duality relation

$$\nabla_{a_1} x_{ij} = \epsilon_{a_1}{}^{b_1 b_2 b_3}\epsilon_{ijk}\nabla_{b_1} x_{b_2 b_3}{}^k\,. \tag{2.7}$$

As with all the duality relations in this paper one can rewrite them so that they contain ∇_α rather than ∇_a using the techniques given at the end of section one. For Eq. (2.7) we find that it can be rewritten as

$$\sqrt{-\gamma}\gamma^{\alpha\beta}\nabla_\beta x_{ij} = \epsilon^{\alpha\beta_1\beta_2\beta_3}\epsilon_{ijk}\nabla_{\beta_1} x_{d_1 d_2}{}^k \nabla_{\beta_2} x^{d_1}\nabla_{\beta_3} x^{d_2} \,. \tag{2.8}$$

Assuming that the other coordinates do not contribute to the dynamics then the number of bosonic degrees of freedom is $7-4=3$ for $x^{a'}$, 3 for x_{ij} and 2 for x_a^k making 8 in all. Thus it contains 6 scalars and one vector as does $N=4$ supersymmetric Yang–Mills theory. We might consider this theory to be a dimensional reduction of the IIB D3 brane.

2.4. *The four brane*

The four brane has a five-dimensional worldvolume and the part of the Lorentz symmetry $SO(1,6)$ which is in the local subalgebra $\mathcal{H}$ is $SO(1,4)\otimes SO(2)$. Examining Eq. (2.1) we find that there are three brane charges with four indices only two of which have four antisymmetric indices. We consider the case that the four brane charge arises from the charge $Z^{\underline{a}_1\cdots\underline{a}_4 M}{}_N$ which belongs to the $(24 = 10\oplus 14)$-dimensional representation of $SO(5)$. We will choose the active brane charge to be $Z^{\underline{a}_1\cdots\underline{a}_4 1}{}_2$ which breaks $SO(5)$ down to $SO(3)$. Under which the 24 contains the four singlets and one of these leads to a coordinate $y^{\underline{a}_1\cdots\underline{a}_4}$ which, together with $x^{\underline{a}}$, obeys Eq. (1.9).

The Lorentz scalar coordinates are dual to the three form coordinates both of which belong to the 10 of $SO(5)$ which decomposes into $10 = 3\oplus 3\oplus 3\oplus 1$ of $SO(3)$. We choose one of the 3's and then write down the generic duality equation

$$\nabla_a x_i = \epsilon_a{}^{b_1\cdots b_4}\nabla_{b_1} x_{b_2 b_3 b_4 i}\,, \quad i = 1,2,3\,. \tag{2.9}$$

The one form coordinates are dual to the two form coordinates and these both belong to the 5 of $SO(5)$. We choose one of the singlets under $SO(3)$ and then we can write down the generic equation

$$\nabla_{[a_1} x_{a_2]} = \epsilon_{a_1 a_2}{}^{b_1 b_2}\nabla_{b_1} x_{b_2}\,. \tag{2.10}$$

We have $7-5=2$ degrees of freedom from the transverse coordinates, 3 from the Lorentz scalars and $5-2=3$ from the one form, which makes a count of 8 bosonic degrees of freedom.

For this case the duality equations are not uniquely determined by the level zero transformations of the local subalgebra and it is quite likely that there are other possible branes corresponding to the different choices of brane charge, the local subalgebra and the selection of different representations of the internal symmetry. What branes actually exist is determined by the full symmetries of the local algebra.

2.5. *The five brane*

The five brane has a six-dimensional worldvolume and the part of the Lorentz symmetry $SO(1,6)$ which is in the local subalgebra $\mathcal{H}$ is $SO(1,5)$. Examining Eq. (2.1) we find that there are five charges with five Lorentz indices which could be the brane charge. We will consider that the active charge is $Z^{\underline{a}_1\cdots\underline{a}_5(MN)}$ and in particular the component $Z^{\underline{a}_1\cdots\underline{a}_5(11)}$ with corresponding coordinate $y_{a_1\cdots a_5}$. As a result the internal $SO(5)$ symmetry is broken to $SO(4)$ which belongs to the local subgroup. As a result $\mathcal{H}_0 = SO(1,5)\otimes SO(4)$. The coordinates $x^{\underline{a}}$ and $y_{\underline{a}_1\cdots\underline{a}_5}$ satisfies the duality relation of Eq. (1.9).

The Lorentz scalar coordinates x_{MN} belong to the 10 of $SO(5)$ which decomposes into the $10 = 6\oplus 4$ of $SO(4)$. It is dual to one of the coordinates with four Lorentz indices. Two of these are $SO(5)$ singlets and the remaining one belongs to the $24 = 10\oplus 14$ of $SO(5)$ which under $SO(4)$ decomposes as $24 = 6\oplus 9\oplus 4\oplus 4\oplus 1$. From these we take one of the 4's and the 4 from the Lorentz scalar coordinates and then we can write down the generic duality relation

$$\nabla_a x_i = \epsilon_a{}^{b_1\cdots b_5}\nabla_{b_1}x_{b_2\cdots b_5 i}\,, \quad i,j = 1,2,3,4\,. \tag{2.11}$$

The two form coordinate $x_{a_1a_2M}$ belongs to the 5 of $SO(5)$ which decomposes into $5 = 4\oplus 1$ under $SO(4)$. Taking the singlet we find the self-duality relation which takes the generic form

$$\nabla_{[a_1}x_{a_2a_3]} = \pm\frac{1}{3!}\epsilon_{a_1a_2a_3}{}^{b_1b_2b_3}\nabla_{b_1}x_{b_2b_3}\,. \tag{2.12}$$

We have $7-6=1$ transverse bosonic degrees of freedom, 4 from the scalars and $\frac{4.3}{2.2} = 3$ from the two form making 8 in all.

For the case of the five brane the level zero transformations of the local subalgebra do not uniquely determine the duality relations. For example we could instead select the six-dimensional representation of $SO(4)$ for the x_{MN} and $x_{b_1\cdots b_4}{}^M{}_N$ and then write down a duality relation. This would contribute 6 bosonic degrees of freedom. We have also assumed that the $x_a{}^M$ and $x_{a_1a_2a_3}{}^{MN}$ do not satisfy a duality relation that leads to degrees of freedom. Indeed, if we had choose each to belong to the four-dimensional representation of $SO(4)$ then we could have written down a duality relation that contributes $4.4 = 16$ degrees of freedom. However, if one wants to get only 8 bosonic degrees of freedom then one must adopt the possibilities we first gave. It is likely that there exist other branes corresponding to different choices and this will be resolved by see which putative brane dynamics carries the full symmetries.

3. Branes in eight dimensions

We will now sketch the dynamics of some of the branes in eight dimensions; the pattern is similar to that in seven dimensions and so we will be brief. The eight-dimensional theory emerges when we decompose E_{11} into $GL(8)$ and the duality symmetry $SL(3)\otimes SL(2)$ which is the algebra that emerges when we delete node eight in the E_{11} Dynkin diagram. The generators in the vector representation are given by

$$
\begin{aligned}
&P_{\underline{a}}\ (1,1);\ Z^{i,j'}\ (3,2);\ Z^{\underline{a}}{}_i\ (\bar{3},1);\ Z^{\underline{a}_1\underline{a}_2 i'}\ (1,2);\\
&\quad Z^{\underline{a}_1\underline{a}_2\underline{a}_3 i}\ (3,1),\ Z^{\underline{a}_1\cdots\underline{a}_4}{}_i{}^{j'}\ (\bar{3},2),\ Z^{\underline{a}_1\cdots\underline{a}_5 i'}{}_{j'}\ (1,3),\ Z^{\underline{a}_1\cdots\underline{a}_5}\ (1,1),\\
&\quad Z^{\underline{a}_1\cdots\underline{a}_5 i}{}_j\ (8,1),\ Z^{\underline{a}_1\cdots\underline{a}_6 i,j'}\ (3,2),\ Z^{\underline{a}_1\cdots\underline{a}_6}{}_{(ij),k'}\ (\bar{6},2),\ldots,
\end{aligned}
\tag{3.1}
$$

where $\underline{a}$, $\underline{b}, \ldots = 0, 1, \ldots, 6$, the numbers in the brackets refer to the representations of $SL(3) \otimes SL(2)$ that the generators belong to and $i, j, \ldots = 1, 2, 3$ and $i', j', \ldots = 1, 2$ are the indices of $SL(3)$ and $SL(2)$, respectively.

As a result the brane moves through a space–time with the coordinates

$$
\begin{aligned}
&x^{\underline{a}}\ (1,1);\ x_{i,j'}\ (\bar{3},2);\ x_{\underline{a}}{}^i\ (3,1);\ x_{\underline{a}_1\underline{a}_2 i'}\ (1,2);\\
&\quad x_{\underline{a}_1\underline{a}_2\underline{a}_3 i}\ (\bar{3},1),\ x_{\underline{a}_1\cdots\underline{a}_4}{}^i{}_{j'}\ (3,2),\ x_{\underline{a}_1\cdots\underline{a}_5 i'}{}^{j'}\ (1,3),\ x_{\underline{a}_1\cdots\underline{a}_5}\ (1,1),\\
&\quad x_{\underline{a}_1\cdots\underline{a}_5 i}{}^j\ (8,1),\ x_{\underline{a}_1\cdots\underline{a}_6 i,j'}\ (\bar{3},2),\ x_{\underline{a}_1\cdots\underline{a}_6}{}^{(ij),k'}\ (6,2),\ldots.
\end{aligned}
\tag{3.2}
$$

The Cartan forms which belong to the vector representation of the E_{11} algebra can be written in the form

$$
\begin{aligned}
\mathcal{V}_l = {}& \nabla x^{\underline{a}} P_{\underline{a}}(1,1) + \nabla x_{i,j'} Z^{i,j'} + \nabla x_{\underline{a}}{}^i Z^{\underline{a}}{}_i\\
&+ \nabla x_{\underline{a}_1\underline{a}_2 i'} Z^{\underline{a}_1\underline{a}_2 i'} + \nabla x_{\underline{a}_1\underline{a}_2\underline{a}_3 i} Z^{\underline{a}_1\underline{a}_2\underline{a}_3 i}\\
&+ \nabla x_{\underline{a}_1\cdots\underline{a}_4}{}^i{}_{j'} Z^{\underline{a}_1\cdots\underline{a}_4}{}_i{}^{j'} + \nabla x_{\underline{a}_1\cdots\underline{a}_5 i'}{}^{j'} Z^{\underline{a}_1\cdots\underline{a}_5 i'}{}_{j'}\\
&+ \nabla x_{\underline{a}_1\cdots\underline{a}_5} Z^{\underline{a}_1\cdots\underline{a}_5} + \nabla Z^{\underline{a}_1\cdots\underline{a}_5 i}{}_j x_{\underline{a}_1\cdots\underline{a}_5 i}{}^j + \cdots.
\end{aligned}
\tag{3.3}
$$

The Cartan forms transform under the local subalgebra $\mathcal{H}$ which is a sublagebra of $I_c(E_{11})$ which at level zero contains the Lorentz algebra $SO(1,7)$ and the Cartan invariant subalgebra of the U duality algebra, that is, $SO(3) \times SO(2)$.

3.1. *The one brane*

The one brane will preserve $SO(1,1) \otimes SO(6)$ of the $SO(1,7)$ Lorentz symmetry. The one brane charge is given by $Z_i^{\underline{a}}$ which belong to the $(3,1)$ representation of the $SO(3) \times SO(2)$. Choosing a given charge we break the internal symmetry $SO(3) \times SO(2)$ down to $SO(2) \times SO(2)$ and we have the decomposition $(3,1) = (2,1) \oplus (1,1)$. Denoting the coordinate associated with the later representation by $y_{\underline{a}}$, we adopt the duality relation of Eq. (1.9) between this coordinate and the usual space–time coordinate $x^{\underline{a}}$.

Examining the coordinates of Eq. (3.2) we find the Lorentz scalar coordinates $x_{i,j'}$ in the $(\bar{3},2)$ representation of the internal symmetry $SO(3) \times SO(2)$; this decomposes into the $(\bar{3},2) = (2,2) \oplus (1,2)$ representations of $SO(2) \times SO(2)$. These must satisfy a self-duality equation and we choose this to hold for the $(2,2)$ representation;

$$\nabla_a x_{ij'} = -\frac{1}{2}\epsilon_a{}^b \epsilon_i{}^k \epsilon_{j'}{}^{l'} \nabla_b x_{kl'}\,, \tag{3.4}$$

where we define, as before, $\nabla_a = (s^{-1})_a{}^\alpha \nabla_\alpha$ and $s_\alpha{}^a = \nabla_\alpha x^a$. In fact, this is not the only equation we can write down which is invariant under the level zero symmetries. We can write the above equation but with no ϵ's in the internal indices and we could also write an equation for the $(1,2)$ representation, also with no ϵ's in the internal indices Using the techniques discussed around Eq. (1.8) we can rewrite Eq. (3.4) in the form

$$\sqrt{-\gamma}\,\gamma^{\alpha\beta}\nabla_\beta x_{ij'} = -\frac{1}{2}\epsilon^{\alpha\beta}\epsilon_i{}^k \epsilon_{j'}{}^{l'} \nabla_\beta x_{kl'}\,. \tag{3.5}$$

Assuming that the equations that follow from the nonlinear realisation imply that these are the only dynamical field we can count the number of bosonic degrees of freedom. We have $8-2=6$ degrees of freedom in $x^{a'}$ and $\frac{2.2}{2}=2$ from $x_{ij'}$ which gives us 8 bosonic degrees of freedom.

3.2. *The two brane*

The two brane preserves only $SO(1,2)\otimes SO(5)$ of the $SO(1,7)$ Lorentz symmetry. The charge $Z^{a_1 a_2 i'}$ of the two brane is which transforms in the $(1,2)$ representation of the internal $SO(3)\times SO(2)$ symmetry and choosing a particular charge, say $Z^{a_1 a_2 1}$, breaks the internal symmetry down to $SO(3)$. As a result $\mathcal{H}_0 = SO(1,2)\otimes SO(4)\otimes SO(3)$. We denote the coordinate associated with the active charge by $y^{\underline{a}_1\underline{a}_2}$ and take this, together with the coordinate $x^{\underline{a}}$, to satisfy the duality relation of Eq. (1.9).

Examining the other coordinates in Eq. (3.2) we find the Lorentz scalar coordinates $x_{ii'}$ in the $(3,2)$ representation of $SO(3)\times SO(2)$. Under the decomposition to $SO(3)$ it breaks into $(3,2)=3\oplus 3$ while the dual coordinate x_a^i belongs to the $(3,1)$ representation of $SO(3)\times SO(2)$ and it, of course, belongs to the 3 of $SO(3)$. As a result we can write the duality equation

$$\nabla_a x_i = \epsilon_a{}^{b_1 b_2}\nabla_{b_1} x_{b_2 i}\,, \quad i = 1,2,3\,. \tag{3.6}$$

Assuming that these are the only active fields the number of bosonic degrees of freedom are $8-3=5$ for $x^{a'}$ and 3 for x_i giving a total of 8 bosonic degrees of freedom.

The two brane in eight dimensions was discussed in Ref. 13 and it would be interesting to see what is the relation to this work.

3.3. *The three brane*

The three brane has a four-dimensional worldvolume and it breaks the Lorentz symmetry $SO(1,7)$ into $SO(1,3) \otimes SO(4)$. The charge for the three brane is $Z^{\underline{a}_1 \underline{a}_2 \underline{a}_3 i}$ which belongs to the $(3,1)$ representation of the internal symmetry $SO(3) \times SO(2)$. Choosing a particular charge preserves only $SO(2) \otimes SO(2)$ and the $(3,1)$ representation becomes the representations $(3,1) = (2,1) \oplus (1,1)$. Denoting the coordinate associated with the last representation by $y_{\underline{a}_1 \underline{a}_2 \underline{a}_3}$, it, together with the coordinate x^a, obeys the duality equation (1.9).

Examining Eq. (3.2) we find the coordinate $x_a{}^i$. This belongs to the $(3,1)$ of $SO(3) \otimes SO(2)$ which decomposes into $(3,1) = (2,1) \oplus (1,1)$. This field is self-dual and choosing the $(2,1)$ representation we can write down the equation

$$\nabla_{[a_1} x_{a_2]i} = \frac{1}{2} \epsilon_{a_1 a_2}{}^{b_1 b_2} \epsilon_i{}^j \nabla_{b_1} x_{b_2 j} \,. \tag{3.7}$$

There is no such consistent equation for the $(1,1)$ representation.

The coordinate $x_{ij'}$ of Eq. (3.2) will be dual to the coordinate $x_{a_1 a_2 i}$ which belong to the $(3,2)$ and $(1,2)$ representations of $SO(3) \times SO(2)$ respectively. These decomposes into the $(3,2) = (1,2) \oplus (2,2)$ and $(1,2) = (1,2)$ representations of $SO(2) \otimes SO(2)$. To write down a duality equation we must choose the in common $(1,2)$ representation and then

$$\nabla_a x_{i'} = \epsilon_a{}^{b_1 b_2 b_3} \nabla_{b_1} x_{b_2 b_3 i'} \,. \tag{3.8}$$

One could include an epsilon in the internal indices if required by the higher level symmetries.

Assuming that the other coordinates do not contribute to the dynamics then the number of bosonic degrees of freedom is $8-4=4$ for $x^{a'}$, 2 for $x_{i'}$ and $4-2=2$ for x_{ai} making 8 in all.

3.4. *The four brane*

The four brane breaks the Lorentz symmetry $SO(1,7)$ into $SO(1,5) \otimes SO(2)$. The charge for the three brane is $Z^{\underline{a}_1 \underline{a}_2 \underline{a}_3 \underline{a}_4}{}_i{}^{j'}$ which belongs to the $(3,2)$ representation of the internal symmetry $SO(3) \times SO(2)$. Choosing a particular charge preserves only $SO(2)$ and the $(3,2)$ representation decomposes as $(3,2) = 2 \oplus 2 \oplus 1 \oplus 1$. Let us denoting one of the singlet coordinates by $y_{\underline{a}_1 \underline{a}_2 \underline{a}_3 \underline{a}_4}$. This coordinate together with the coordinate x^a, obeys the duality equation (1.9).

The Lorentz scalar coordinates $x_{ij'}$ belong to the $(3,2)$ representation of $SO(3) \times SO(2)$ and which decomposes as given above. This coordinate is dual to the three form coordinate $x_{\underline{a}_1 \underline{a}_2 \underline{a}_3 i}$ which belongs to the $(3,1)$ representation of $SO(3) \times SO(2)$. The two representations have in common the 2 and 1 representations If we choose the former representation we can write down the duality relation

$$\nabla_a x_i = \epsilon_a{}^{b_1 b_2 b_3 b_4} \epsilon^{ij} \nabla_{b_1} x_{b_2 b_3 b_4 j} \,, \quad i,j = 1,2 \,. \tag{3.9}$$

In fact one could omit the epsilon in the internal indices and we could also taken instead the singlet at the symmetry level at which we are working.

The one form coordinate $x_a{}^i$ is dual to the two form coordinate $x_{a_1 a_2 i'}$ which belong to the $(3,1)$ and $(1,2)$ representation of $SO(3) \times SO(2)$ respectively. When decomposed into $SO(2)$ they have in common only a singlet of $SO(2)$ and we can write down the generic equation

$$\nabla_{[a_1} x_{a_2]} = \epsilon_{a_1 a_2}{}^{b_1 b_2 b_3} \nabla_{b_1} x_{b_2 b_3} \,. \tag{3.10}$$

We have $8-5=3$ transverse bosonic degrees of freedom, 2 in x_i and $(5-2)=3$ in x_a, making 8 in all.

4. The One Brane in Four Dimensions

We now briefly discuss the one brane in four dimensions. Decomposing E_{11} into representations of $GL(4) \otimes E_7$ we find the theory in four dimensions. The level zero part of $I_c(E_{11})$ is $SO(1,3) \otimes SU(8)$. The coordinates at low levels can be read off from the table given earlier in this paper. It will be useful to present the coordinates in terms of representations of $SU(8)$

$$x^{\underline{a}}(1),\ x^{ij}(28),\ x_{ij}(\bar{2}8),\ x^{ai}{}_j(63),\ x^{ai_1\cdots i_4}(70),\ x^a, \ldots, (1)\,, \quad i,j,\ldots = 1,\ldots,8\,, \tag{4.1}$$

where the number in the brackets gives the dimensions of the $SU(8)$ representations.

The one brane preserves $SO(1,1) \otimes SO(2)$ of the $SO(1,3)$ Lorentz symmetry. The brane charge belongs to the $63 \oplus 70 \oplus 1$ representations of $SU(8)$. Let us choose the it to belong to the 70-dimensional representation, that is, $Z^{ai_1\cdots i_4}$ and take the charge Z^{a1234} to be the active charge. As a result the internal symmetry $SU(8)$ gets broken to $SU(4) \otimes SU(4)$ with the decomposition $70 = (1,1) \oplus (1,1) \oplus (\bar{4},4) \oplus (6,6) \oplus (4,\bar{4})$ with Z^{a1234} being one of the singlets. We denote the corresponding coordinate by $y_{\underline{a}}$ and it, together with $x^{\underline{a}}$ will obey Eq. (1.9).

The Lorentz scalar coordinates decompose into representations of $SU(4) \otimes SU(4)$ as $28 = (6,1) \oplus (4,4) \oplus (1,6)$ with similar results for the $\overline{28}$. These coordinates must obey a self-duality condition, namely

$$\nabla_a x^{ij} = \frac{1}{2} \epsilon_a{}^b \epsilon^{ijkl} \nabla_b x_{kl}\,, \quad i,j,k,l = 1,2,3,4\,. \tag{4.2}$$

Counting the bosonic degrees of freedom we have $4-2=2$ from the transverse coordinates $x^{a'}$ and $\frac{4.3}{2} = 6$ from the x^{ij} making 8 in all. There may well be other possible one branes one could construct using the full symmetries of the nonlinear realisation.

5. Discussion

In two previous papers[6,7] we discussed how to construct brane dynamics as a nonlinear realisation of $E_{11} \otimes_s l_1$ and we have shown that it leads to many features of the brane dynamics that we know. While the construction of the dynamics of a given brane is rather complicated there have emerged a number of generic features;

for example the dynamical equations when constructed from the Cartan forms are a set of duality equations. By construction these equations are invariant under the symmetries of the nonlinear realisation and in particular the lowest level such symmetries. In this paper we have applied these general features to find *in outline only* the dynamics of the low level branes in seven and eight dimensions. We find that the coordinates of the vector representation do indeed provided the fields for a set of duality relations that look to be of the right type in that they contain the expected coordinates describing how the brane moves through the usual space–time as well as the required worldvolume fields. Unlike the superficial impression that might be gained by studying the familiar branes in eleven and ten dimensions, worldvolume fields are generically present as long as they are consistent with the duality relations of Eq. (1.10). Hence although this paper contains no calculations of any length we hope that it does provide an insight into the general form of the dynamics of branes in E theory that is not obscured by the formalism or lengthy equations.

We find that the generic form of the dynamical equations are generally determined by the lowest level symmetries although for some of the higher level branes considered here are several possibilities. It would be interesting to see how the generic dynamical equations become completely determined once they are required to be invariant under the higher level symmetries.

While Refs. 6 and 7 set out the general method to determine the brane dynamics from the nonlinear realisation there are a number of steps where a very systematic path is absent. The situation is not unlike that which occurred when the $E_{11} \otimes_s l_1$ nonlinear realisation was used to find the low energy effective action for strings and branes, indeed it took quite a few years before the unique path became clear and the dynamics constructed. Three of the outstanding issues are as follows:

- The nonlinear realisation requires for its construction a choice of local subalgebra $\mathcal{H}$ which for the branes is a subalgebra of the Cartan involution invariant subalgebra of E_{11}. Studies of particular branes have shown that the local subalgebras are much more subtle than one might naively expect.[7] It would be good to have a systematic way of choosing the local subalgebras that lead to brane dynamics; indeed such an understanding may lead to a classification of all branes from a purely algebraic viewpoint. We hope to report on progress in this direction elsewhere.
- The brane dynamics, as usually formulated, consists of equations that contain second order derivatives acting on some of the fields. However, the brane dynamics that emerges from E theory is a set of duality equations that are first order in derivatives acting on fields. This is like the derivation of the low energy effective action of strings and branes from E theory. In the later case one can act on the duality equations with a space–time derivative to eliminate field certain fields and obtain equations which are second order in derivatives and are those of maximal supergravity once one restricts the equations to the lowest level fields and coordinates. One would expect that a similar pattern will hold for the brane

dynamics derived from E theory and it would be good to see that this is the case. In fact this is the case for branes with no worldvolume fields and at the linearised level in worldvolume fields when they are present. However, it would be good to see the known features of the worldvolume fields emerge at the nonlinear level.

The equations that follow from the nonlinear realisation contain dynamical equations, as studied in this paper, and algebraic equations that solve for some of the φ fields, associated with the breaking of $I_c(E_{11})$ to $\mathcal{H}$, in terms of derivatives acting on the coordinates. However, the dynamical equations also contain the φ fields and so only once one has solved for these fields can on see the final form of the dynamical equation in terms of the usual fields. A systematic way to do this has yet to be found.

- The nonlinear realisation for branes is formulated so that the fields and coordinates depend on the brane parameters which label points in the worldvolume swept out by the brane. However, in E theory the space–time is infinite-dimensional and so one can wonder what is the brane worldvolume in this very large space–time?

The vector representations at low levels contains the brane charges of all branes that we are familiar with. However, it contains an infinite number of branes and it is reasonable to suppose that it encodes all brane charges. As a result E theory contains an infinite number of new degrees of freedom which might be very useful when studying problem such as black hole entropy. Once the above issues are resolved it would be very interesting to determine the dynamics of the exotic branes which occur at higher levels in the vector representation.

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